ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT
Appl. Comput. Harmon. Anal. 20 (2006) 149–157
Applied and Computational Harmonic Analysis
www.elsevier.com/locate/acha

Letter to the Editor

Some properties of spectral measures ☆

Izabella Łaba [a,*], Yang Wang [b]

[a] *Department of Mathematics, University of British Columbia, Vancouver, Canada, B.C. V6T 1Z2*
[b] *School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA*

Available online 27 July 2005

Communicated by Akram Aldroubi on 2 November 2004

Abstract

A Borel measure μ in $\mathbb{R}^d$ is called a *spectral measure* if there exists a set $\Lambda \subset \mathbb{R}^d$ such that the set of exponentials $\{\exp(2\pi i\lambda \cdot x)\colon \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\mu)$. In this letter we prove some properties of spectral measures. In particular, we prove results that highlight the $3/2$-rule.

1. Introduction

Spectral measures, first introduced by Jorgensen and Pedersen [4], are a natural extension of spectral sets. Let μ be a finite Borel measure in $\mathbb{R}^d$. We say μ is a *spectral measure* if there exists a set $\Lambda \subset \mathbb{R}^d$ such that the set of exponentials $\{\exp(2\pi i\lambda \cdot x)\colon \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\mu)$. In this case we call Λ a *spectrum* of μ, and (μ, Λ) a *spectral pair*. It should be pointed out that a measure μ may have more than one spectrum, see, e.g., Łaba and Wang [7]. Let Ω be a measurable set in $\mathbb{R}^d$ and $\mu = m|_\Omega$, the restriction of the d-dimensional Lebesgue measure m to Ω. We say Ω is a *spectral set* if μ is a spectral measure.

☆ The first author is partially supported by the Natural Science and Engineering Research Council of Canada. The second author is supported in part by the National Science Foundation Focused Research Group grant DMS 0139261.
* Corresponding author.
E-mail addresses: ilaba@math.ubc.ca (I. Łaba), wang@math.gatech.edu (Y. Wang).

1063-5203/$ – see front matter
doi:10.1016/j.acha.2005.03.003

The main interest for studying spectral sets comes from its mysterious connection to tiling, first formulated in a conjecture by B. Fuglede [2], known today as the Fuglede Conjecture or the Spectral Set Conjecture.

The Fuglede Conjecture. A measurable set Ω in $\mathbb{R}^d$ is a spectral set if and only if it tiles $\mathbb{R}^d$ by translation.

The conjecture had baffled the mathematicians who studied spectral sets for years until very recently, when Tao [12] exhibited a spectral set in dimensions $d \geqslant 5$ that is not a tile, and Kolountzakis and Matolsci [5] exhibited tiles that are not spectral sets in dimensions $d \geqslant 5$. Despite the counterexamples, the connection between spectral sets and tiling is strongly evident, especially in low dimensions, as indicated by earlier works, including the original work of Fuglede [2]. Many positive results have been established as well, see, e.g., Lagarias and Wang [8], Pedersen and Wang [10], Łaba [6], and Iosevich, Katz, and Tao [3]. Interestingly, there is even evidence showing a strong connection between tiling and spectral measures, see Łaba and Wang [7] and Strichartz [11].

The study of spectral measures so far has focused on self-similar measures. These measures have the advantage that their Fourier transforms can be explicitly written down as an infinite product, which allows us to compute their zeros. In this letter, we examine measures that are not self-similar. Without knowing the set of zeros of their Fourier transforms, the characterization of spectral measures becomes difficult. Here we focus on some fundamental properties of spectral measures. While the results in this letter are modest, they are rather general and should offer some valuable guidance to future studies in this area.

A Borel measure in $\mathbb{R}^d$ is *discrete* if it is supported on a countable set. It is well known that any Borel measure μ in $\mathbb{R}^d$ can be decomposed uniquely as $\mu = \mu_c + \mu_d$, where μ_d is a discrete measure and μ_c has no "atoms," i.e. $\mu_c(\{x\}) = 0$ for any singleton $\{x\}$ in $\mathbb{R}^d$. We say that μ has no discrete part if $\mu_d = 0$. We prove:

Theorem 1.1. *Let μ be a spectral Borel measure in $\mathbb{R}^d$. Then either μ is discrete, or it has no discrete part.*

Note that μ has a Lebesgue decomposition $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous and μ_s is singular with respect to the Lebesgue measure m. All known examples of spectral measure on $\mathbb{R}^d$ are either absolutely continuous or purely singular. We do not know whether this must always be the case.

Theorem 1.2. *Let μ be a nonzero finite spectral Borel measure in $\mathbb{R}^d$.*

(1) *Assume that μ is absolutely continuous. Then* $\mathrm{supp}(\mu)$ *has positive and finite Lebesgue measure. Moreover, if μ has a density function $p(x)$ satisfying*

$$\left|\hat{p}(\xi)\right| \leqslant C\left(1 + |\xi|\right)^{-\alpha}, \quad \alpha > (d+1)/2 \tag{1.1}$$

(*this holds, e.g., when $p(x)$ is smooth*), *then μ cannot be spectral.*

(2) *Suppose that μ is discrete, $\mu = \sum_{a \in A} p_a \delta_a$, where $A \subset \mathbb{R}^d$, δ_a is the point mass at a and $p_a > 0$. Then A is a finite set, and all p_a are equal.*

The exponent $(d+1)/2$ in (1.1) may be a natural threshold, as well known stationary phase estimates show that the Fourier transform of a characteristic function of a convex body whose boundary is smooth and has nonvanishing Gaussian curvature has exactly this rate of decay. It may well be true that the density of a spectral measure should be constant on its support, but proving this may require new methods.

Finally we establish two results that highlight the 3/2-rule.

Theorem 1.3. *Let $\mu = \sum_{j=0}^{N-1} p_j \delta_{a_j}$ be a discrete spectral measure in $\mathbb{R}$, where $\{a_j\} \subset \mathbb{Z}$, $N \geqslant 1$. Assume that* $\max_j a_j - \min_j a_j < 3N/2 - 1$. *Then* $\{a_j\} \pmod N = \{0, 1, \ldots, N-1\}$.

We point out that if $\{a_j\colon 0 \leqslant j < N\} \pmod N = \{0, 1, \ldots, N-1\}$ then $\mu = \sum_{j=0}^{N-1} \delta_{a_j}$ is a spectral measure, with a spectrum $\Lambda = \{j/N\colon 0 \leqslant j < N\}$.

Theorem 1.4. *Let μ be an absolutely continuous spectral measure in $\mathbb{R}$ with* $\operatorname{supp}(\mu) = \Omega$. *Let $m(\Omega) = a$ and* $\operatorname{diam}(\Omega) = \Delta$. *Suppose that $\Delta < 3a/2$. Then Ω tiles $\mathbb{R}$ by translation by the lattice $a\mathbb{Z}$, and μ has density $p(x) = c\chi_\Omega$ for some $c > 0$. In particular if Ω is an interval then $p(x) = c\chi_\Omega$ for some $c > 0$.*

These theorems will be proved in the rest of the letter. The proof of the last two theorems depends on a combinatorial result concerning the cardinality and asymptotic density of $A - A$ for a given set A in $\mathbb{R}$, which we prove in Section 2.

2. A combinatorial result

In this section we establish a combinatorial result that is the key to our theorems related to the 3/2-rule. First we introduce the asymptotic density of a set. Let $A \subseteq \mathbb{R}^d$. The *lower* and *upper asymptotic density* of A are given respectively by

$$D_-(A) = \liminf_{N\to\infty} \frac{\#(A \cap [-N, N]^d)}{2^d N^d}, \qquad D_+(A) = \limsup_{N\to\infty} \frac{\#(A \cap [-N, N]^d)}{2^d N^d}.$$

If $D_-(A) = D_+(A)$ then we denote them by $D(A)$.

Lemma 2.1. *For any $A, B \subseteq \mathbb{R}^d$ we have $D_-(A \cap B) \geqslant D_-(A) + D_-(B) - D_+(A \cup B)$ and $D_+(A \cap B) \geqslant D_-(A) + D_-(B) - D_-(A \cup B)$.*

Proof. Let ν be the counting measure in $\mathbb{R}^d$. Then

$$\begin{aligned}
D_-(A) + D_-(B) &= \liminf_{N\to\infty} \frac{1}{2^d N^d} \int_{[-N,N]^d} \chi_A \,\mathrm{d}\nu + \liminf_{N\to\infty} \frac{1}{2^d N^d} \int_{[-N,N]^d} \chi_B \,\mathrm{d}\nu \\
&\leqslant \liminf_{N\to\infty} \frac{1}{2^d N^d} \int_{[-N,N]^d} (\chi_A + \chi_B) \,\mathrm{d}\nu = \liminf_{N\to\infty} \frac{1}{2^d N^d} \int_{[-N,N]^d} (\chi_{A\cup B} + \chi_{A\cap B}) \,\mathrm{d}\nu \\
&\leqslant \limsup_{N\to\infty} \frac{1}{2^d N^d} \int_{[-N,N]^d} \chi_{A\cup B} \,\mathrm{d}\nu + \liminf_{N\to\infty} \frac{1}{2^d N^d} \int_{[-N,N]^d} \chi_{A\cap B} \,\mathrm{d}\nu \\
&= D_+(A \cup B) + D_-(A \cap B).
\end{aligned}$$

This proves the first inequality. Note that in the last inequality above one can switch the positions of the $\liminf$ and $\limsup$. Making the switch yields the second inequality of the lemma. □

For any $A \in \mathbb{R}^d$ we denote $A - A = \{a - a' \colon a, a' \in A\}$. Our next result concerns the density of $A - A$. Part (1) of it is a refinement of an argument in [6].

Proposition 2.2. (1) *Let G be an Abelian group and $A \subseteq G$ with $|A| = N$. Suppose that $|A - A| < 3N/2$. Then $A - A$ is a subgroup of G.*

(2) *Let $A \subset \mathbb{R}^d$. Assume that $D_+(A - A) < 3/2D_-(A)$. Then $A - A$ is a group.*

Proof. (1) We prove that $A - A$ is closed under subtraction, which implies that $A - A$ is a subgroup of G. For any $x, y \in A - A$ we show that there exist a, b, c in A such that $x = a - c$ and $y = b - c$. To see this denote $B_x := (A - x) \cap A$ and $B_y := (A - y) \cap A$. Write $x = a_1 - a_2$, where $a_1, a_2 \in A$. We observe that $(A - x) \cup A = (A - \{a_1, a_2\}) + a_2$. Hence its cardinality cannot exceed the cardinality of $A - A$, which is less than $3N/2$. It follows that

$$|B_x| = |A - x| + |A| - \bigl|(A - x) \cup A\bigr| > N + N - \frac{3N}{2} = \frac{N}{2}.$$

Similarly $|B_y| > N/2$. Since $B_x \cup B_y \subseteq A$, we have

$$|B_x \cap B_y| > N/2 + N/2 - N > 0.$$

Now let $c \in B_x \cap B_y$. So $c \in A$, and furthermore, $c = a - x = b - y$ for some $a, b \in A$. Thus $x - y = a - b \in A - A$, proving that $A - A$ is closed under subtraction.

(2) We prove that $A - A$ is closed under subtraction almost verbatim as in (1). Without loss of generality we assume that $D_-(A) = 1$ and $D_+(A - A) < 3/2$. For any $x, y \in A - A$ we show that there exist a, b, c in A such that $x = a - c$ and $y = b - c$. Again denote $B_x := (A - x) \cap A$ and $B_y := (A - y) \cap A$. Write $x = a_1 - a_2$, where $a_1, a_2 \in A$. We observe that $(A - x) \cup A = (A - \{a_1, a_2\}) + a_2$. Hence $D_+((A - x) \cup A) < 3/2$. It follows from Lemma 2.1 that

$$D_-(B_x) \geqslant D_-(A - x) + D_-(A) - D_+\bigl((A - x) \cup A\bigr) > \frac{1}{2}.$$

Similarly $D_-(B_y) > 1/2$. Since $B_x \cup B_y \subseteq A$, again by Lemma 2.1

$$D_+(B_x \cap B_y) \geqslant D_-(B_x) + D_-(B_y) - D_-(A) > 0.$$

Hence $B_x \cap B_y$ is nonempty.

Now let $c \in B_x \cap B_y$. So $c \in A$, and furthermore, $c = a - x = b - y$ for some $a, b \in A$. Thus $x - y = a - b \in A - A$, proving that $A - A$ is closed under subtraction. □

3. Proof of theorems

Throughout this section we shall let $e_\lambda(x) := \exp(2\pi i \lambda \cdot x)$ and $\langle f, g\rangle_\mu = \int_{\mathbb{R}^d} f \bar{g} R \, \mathrm{d}\mu$. We also use $L^2(\Omega)$ to denote the L^2 space with respect to the Lebesgue measure on Ω, and write $\langle f, g\rangle = \int_{\mathbb{R}^d} f \bar{g} \, \mathrm{d}x$.

Proof of Theorem 1.1. Without loss of generality we assume that μ is a probability measure, i.e. $\mu(\mathbb{R}^d) = 1$. Let $\mu = \mu_c + \mu_d$, where μ_d is the discrete part of μ. We prove that either $\mu_c = 0$ or $\mu_d = 0$.

Assume otherwise. Let $\mu_d(\{x_0\}) = p > 0$. Consider the function $f = \chi_{\{x_0\}}$. Clearly $f \in L^2(\mu)$. Let Λ be a spectrum for μ. Note that

$$\langle f(x), e_\lambda(x)\rangle = \int_{\mathbb{R}^d} f(x)e_\lambda(-x)\,\mathrm{d}\mu(x) = pf(x_0)e_\lambda(-x_0).$$

It follows from Parseval's equality that

$$\|f\|^2_{L^2(\mu)} = \sum_{\lambda\in\Lambda} \bigl|\langle f(x), e_\lambda(x)\rangle_\mu\bigr|^2 = \sum_{\lambda\in\Lambda} p^2 < \infty. \tag{3.1}$$

Hence Λ is a finite set. But if so then $L^2(\mu)$ is finite dimensional, which is not the case since $\mu_c \neq 0$. This is a contradiction. □

To prove our next theorem we will be using some well-known results on the density of sampling and interpolations of band-limited functions. Let

$$\mathcal{B}(\Omega) := \bigl\{\hat{f}\colon f \in L^2(\Omega)\bigr\}.$$

We say that $\Lambda \subseteq \mathbb{R}^d$ is a *sampling* of $\mathcal{B}(\Omega)$ if there exists a constant $C > 0$ such that for every $f \in L^2(\Omega)$ we have

$$\sum_{\lambda\in\Lambda} \bigl|\langle f, e_\lambda\rangle\bigr|^2 \geqslant C\|f\|_2^2.$$

We say that $\Lambda \subseteq \mathbb{R}^d$ is an *interpolation* of $\mathcal{B}(\Omega)$ if for every $\{c_\lambda\} \in l^2(\Lambda)$ there exists an $f \in L^2(\Omega)$ such that $c_\lambda = \langle f, e_\lambda\rangle$. A theorem of H. Landau [9] states that $D_-(\Lambda) \geqslant m(\Omega)$ for a sampling Λ of $\mathcal{B}(\Omega)$, and $D_+(\Lambda) \leqslant m(\Omega)$ for an interpolation Λ of $\mathcal{B}(\Omega)$.

Proof of Theorem 1.2. We first prove part (2). Let $\mu = \sum_{a\in A} p_a\delta_a$, where A is a finite or countable set in $\mathbb{R}^d$, $p_a > 0$ for all $a \in A$. Without loss of generality we assume μ is a probability measure, i.e. $\sum_{a\in A} p_a = 1$. Let Λ be a spectrum of μ. Fix a $a_0 \in A$ and let $f = \chi_{\{a_0\}}$. By (3.1) we have

$$\sum_{\lambda\in\Lambda} p_a^2 = \|f\|^2_{L^2(\mu)} = p_a.$$

Thus $p_a \cdot \#\Lambda = 1$. This proves part (2) of the theorem.

Now we prove part (1). It is clear that $\Omega = \operatorname{supp}(\mu)$ has positive Lebesgue measure. We prove that the measure is also finite. Let $p(x)$ be the density function of μ and Λ be a spectrum for μ. For each $N > 0$ define

$$\Omega_N := \left\{x\colon x \in \Omega \cap [-N, N]^d,\ \frac{1}{N} \leqslant p(x) \leqslant N\right\}.$$

We prove that Λ is a sampling for $\mathcal{B}(\Omega_N)$. For any $f \in L^2(\Omega_N)$ set $\tilde{f}(x) = f(x)/p(x)$ for $x \in \Omega_N$ and $\tilde{f}(x) = 0$ otherwise. Then

$$\int_{\mathbb{R}^d} |\tilde{f}|^2\,\mathrm{d}\mu = \int_{\Omega_N} \bigl|f(x)\bigr|^2 p^{-1}(x)\,\mathrm{d}x \leqslant N\int_{\Omega_N} \bigl|f(x)\bigr|^2\,\mathrm{d}x < \infty.$$

Hence $\tilde{f} \in L^2(\mu)$. We have

$$\int_{\mathbb{R}^d} |\tilde{f}|^2 \, d\mu = \sum_{\lambda \in \Lambda} \left| \langle \tilde{f}, e_\lambda \rangle \right|^2 = \sum_{\lambda \in \Lambda} \left| \int_{\Omega_N} f(x) \overline{e_\lambda(x)} \, dx \right|^2.$$

It follows from the inequality

$$\int_{\mathbb{R}^d} |\tilde{f}|^2 \, d\mu \geqslant \frac{1}{N} \int_{\Omega_N} \left| f(x) \right|^2 dx$$

that Λ is a sampling for $\mathcal{B}(\Omega_N)$.

Now the theorem of Landau [9] yields $D_-(\Lambda) \geqslant m(\Omega_N)$, which implies

$$D_-(\Lambda) \geqslant m(\Omega) \tag{3.2}$$

by letting $N \to \infty$. If $m(\Omega) = \infty$, it follows that for any $\varepsilon > 0$ there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $|\lambda_1 - \lambda_2| < \varepsilon$. However by choosing ε sufficiently small we have $\int_{\mathbb{R}^d} e_{\lambda_1 - \lambda_2} \, d\mu \neq 0$, contradicting the orthogonality of $\{e_\lambda \colon \lambda \in \Lambda\}$. Therefore Ω must have finite Lebesgue measure.

We note for future reference that the last paragraph implies that

$$|\lambda - \lambda'| \geqslant \varepsilon > 0 \quad \text{if } \lambda, \lambda' \in \Lambda, \ \lambda \neq \lambda'. \tag{3.3}$$

We now prove the second claim in (2). If Λ is a spectrum for μ, then for any $f \in L^2(\mu)$ we must have

$$\|f\|^2_{L^2(\mu)} = \sum_{\lambda \in \Lambda} \left| \langle f, e_\lambda \rangle_\mu \right|^2.$$

In particular, setting $f(x) = e_\xi(x)$ we get

$$1 = \|e_\xi\|^2_{L^2(\mu)} = \sum_{\lambda \in \Lambda} \left| \int e_{\xi - \lambda}(x) p(x) \, dx \right|^2 = \sum_{\lambda \in \Lambda} \left| \hat{p}(\xi - \lambda) \right|^2. \tag{3.4}$$

Next, we claim that this and the assumption (1.1) imply that

$$D_+(\Lambda) \leqslant \|p\|_2^{-2}. \tag{3.5}$$

Assuming (3.5), we complete the proof of the theorem as follows. By Cauchy–Schwarz, we have

$$1 = \int_\Omega p(x) \, dx \leqslant \left(\int p^2(x) \, dx \right)^{1/2} \cdot \left(\int_\Omega 1 \right)^{1/2} = m(\Omega)^{1/2} \|p\|_2. \tag{3.6}$$

Combining this with (3.2) and (3.5), we get

$$m(\Omega) \leqslant D_-(\Lambda) \leqslant D_+(\Lambda) \leqslant \|p\|_2^{-2} \leqslant m(\Omega).$$

Thus all inequalities above must in fact be equalities. In particular, we must have an equality in (3.6), which is possible if and only if $p(x)$ is a constant function on Ω.

It remains to prove (3.5). Let N be a large number and let $Q_N = [-N, N]^d$, then from (3.4) we have

$$(2N)^d = \sum_{\lambda \in \Lambda} \int_{Q_N} \left| \hat{p}(\xi - \lambda) \right|^2 d\xi = \sum_{\lambda \in \Lambda \cap Q_N} \int_{\mathbb{R}^d} \left| \hat{p}(\xi - \lambda) \right|^2 d\xi - \sum_{\lambda \in \Lambda \cap Q_N} \int_{\mathbb{R}^d \setminus Q_N} \left| \hat{p}(\xi - \lambda) \right|^2 d\xi$$

$$+\sum_{\lambda\in\Lambda\setminus Q_N}\int_{Q_N}\left|\hat{p}(\xi-\lambda)\right|^2 \mathrm{d}\xi =: \mathrm{I}-\mathrm{II}+\mathrm{III}.$$

Hence

$$\#(\Lambda\cap Q_N)\cdot\|p\|_2^2=\mathrm{I}=(2N)^d+\mathrm{II}-\mathrm{III}\leqslant(2N)^d+\mathrm{II},$$

and (3.5) will follow if we show that $\mathrm{II}=o(N^d)$, which we now proceed to do. Consider those $\lambda\in\Lambda\cap Q_N$ with $\operatorname{dist}(\lambda,\partial Q_N)\leqslant\log N$. By (3.3), the number of such λ is $O(N^{d-1}\log N)$, hence

$$\sum_{\lambda\in\Lambda\cap Q_N:\operatorname{dist}(\lambda,\partial Q_N)\leqslant\log N}\ \int_{\mathbb{R}^d\setminus Q_N}\left|\hat{p}(\xi-\lambda)\right|^2 \mathrm{d}\xi=O\big(N^{d-1}\log N\big). \tag{3.7}$$

Now, consider those $\lambda\in\Lambda\cap Q_N$ with $\operatorname{dist}(\lambda,\partial Q_N)\in[2^j\log N,2^{j+1}\log N)$, where j is a nonnegative integer. By (3.3) again, the number of such λ is $O(2^j\log N\cdot N^{d-1})$. For each such λ, we have

$$\int_{\mathbb{R}^d\setminus Q_N}\left|\hat{p}(\xi-\lambda)\right|^2 \mathrm{d}\xi\leqslant C\int_{\|\xi\|\geqslant 2^j\log N}|\xi|^{-2\alpha}\,\mathrm{d}\xi\leqslant C\big(2^j\log N\big)^{d-2\alpha},$$

by (1.1). Thus the total contribution for a fixed j is bounded by

$$O\big(2^j\log N\cdot N^{d-1}\cdot\big(2^j\log N\big)^{d-2\alpha}\big)=O\big(N^{d-1}(\log N)^{d-2\alpha+1}\cdot 2^{j(d-2\alpha+1)}\big).$$

Summing over j, we bound II by

$$O\big(N^{d-1}\log N\big)+O\left(N^{d-1}(\log N)^{d-2\alpha+1}\cdot\sum_{j=0}^{\infty}2^{j(d-2\alpha+1)}\right)=O\big(N^{d-1}\log N\big),$$

since the series is convergent if $\alpha>(d+1)/2$. This completes the proof of (3.5) and of the theorem. □

Proof of Theorem 1.3. Since the translation of a spectral measure is again a spectral measure with the same spectra, we may without loss of generality assume that $0=a_0<a_1<a_2<\cdots<a_{N-1}=M$. Also, because all p_j are the same by Theorem 1.2(2), we may assume that $p_j=1$ for all j and $\mu=\sum_{j=0}^{N-1}\delta_{a_j}$. Now let Λ be a spectrum of μ, $|\Lambda|=N$. Then all nonzero elements of $\Lambda-\Lambda$ are contained in the zero set of $\hat{\mu}(\xi)=\sum_{j=0}^{N-1}\exp(2\pi i a_j\xi)$. Note that if λ is a zero of $\hat{\mu}$ then so is $\lambda+k$ for any $k\in\mathbb{Z}$. Hence we may view the zeros of $\hat{\mu}$ as elements in the group $\mathbb{T}$.

By assumption, $M=a_{N-1}<3N/2-1$, which implies that $\hat{\mu}(\xi)$ has no more that $3N/2-1$ roots in $\mathbb{T}$. Therefore the cardinality of $\Lambda-\Lambda$ viewed as a subset of $\mathbb{T}$ is no more than $M+1<3N/2$. It follows from Lemma 2.2 that $\Lambda-\Lambda$ is a subgroup of $\mathbb{T}$. Hence viewed as elements in $\mathbb{T}$, $\Lambda-\Lambda=\{j/K\colon 0\leqslant j<K\}$ for some positive integer $K\geqslant N$. This means $\hat{\mu}(j/K)=0$ for all $1\leqslant j<K$. Let $q(z)=\sum_{j=0}^{N-1}z^{a_j}$. It follows that all Kth roots of unity $\neq 1$ are roots of $q(z)$, and $1+z+\cdots+z^{K-1}|q(z)$.

Now let $a_j\equiv b_j \pmod K$, where $0\leqslant b_j<K$. Then $z^K-1|z^{a_j}-z^{b_j}$. Hence $1+z+\cdots+z^{K-1}|\sum_{j=0}^{N-1}z^{b_j}$. But this is impossible if $N<K$. Thus we have $K=N$. Furthermore $\{b_j\}=\{0,1,\ldots,N-1\}$. This proves the theorem. □

Proof of Theorem 1.4. The direction $\Leftarrow$ is obvious. We prove the $\Rightarrow$ direction.

Let $p(x) \in L^2(\mathbb{R})$ be the density function of μ, and let Λ be a spectrum of μ. For any $N > 0$ denote

$$\Omega_N := \left\{x\colon\ x \in \Omega,\ \frac{1}{N} \leqslant p(x) \leqslant N\right\}.$$

The same argument from the proof of Theorem 1.2 shows Λ is a sampling for $\mathcal{B}(\Omega_N)$. This yields $D_-(\Lambda) \geqslant m(\Omega) = a$, using the theorem of Landau [9] and letting $N \to \infty$.

We next prove that $\Lambda - \Lambda = b\mathbb{Z}$ for some $b > 0$. First we show it is a group. Observe that $\Lambda - \Lambda$ is contained in the set

$$Z_p := \left\{\xi \in \mathbb{R}\colon\ \hat{p}(\xi) = 0\right\} \cup \{0\}.$$

Since $\hat{p}$ is an entire function of exponential type with the diameter of supp $p = \Omega$ being less than $(3/2)a$, it is well known [1] that

$$D_+(Z_p) \leqslant \operatorname{diam}(\Omega) < \frac{3}{2}a.$$

This yields $D_+(\Lambda - \Lambda) < (3/2)a$. Proposition 2.2 now implies that $\Lambda - \Lambda$ is a group. Since $\Lambda - \Lambda$ is discrete and the only discrete subgroups of $\mathbb{R}$ are cyclic groups, $\Lambda - \Lambda = b\mathbb{Z}$ for some $b > 0$. Furthermore, Λ is "maximal" in the sense that one cannot add another element to it so that $\Lambda - \Lambda = b\mathbb{Z}$ is not violated as a result of orthonormal basis. Thus $\Lambda = b\mathbb{Z} + \lambda_0$. Since a translate of a spectrum is also a spectrum, we may assume that $\lambda_0 = 0$.

It remains to prove that $b = a^{-1}$. Notice that $b\mathbb{Z} \setminus \{0\} \subseteq Z_p$ implies that

$$\sum_{n\in\mathbb{Z}} p\left(x - b^{-1}n\right) \equiv c \quad \text{a.e. } x \in \mathbb{R}. \tag{3.8}$$

Hence p is bounded. We prove that Λ is an interpolation for $\mathcal{B}(\Omega)$. For any $\{c_\lambda\} \in l^2(\Lambda)$ there exists an $f \in L^2(\mu)$ such that $\int_{\mathbb{R}} f\overline{e_\lambda}\,\mathrm{d}\mu = c_\lambda$ for all $\lambda \in \Lambda$. Since p is bounded $g = fp \in L^2(\Omega)$. But $\int_\Omega fp\overline{e_\lambda}\,\mathrm{d}x = c_\lambda$. Therefore Λ is an interpolation for $\mathcal{B}(\Omega)$. It follows from Landau's theorem that $D_+(\Lambda) \leqslant a$. Hence

$$D(\Lambda) = D_+(\Lambda) = D_-(\Lambda) = a.$$

Finally, (3.8) combines with $m(\Omega) = a$ to yield $p = c\chi_\Omega$, and Ω tiles by $a\mathbb{Z}$.

References

[1] R.P. Boas, Entire Functions, Academic Press, New York, 1954.
[2] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, J. Funct. Anal. 16 (1974) 101–121.
[3] A. Iosevich, N. Katz, T. Tao, Fuglede conjecture holds for convex planar domains, Math. Res. Lett. 10 (2003) 559–569.
[4] P.E.T. Jorgensen, S. Pedersen, Dense analytic subspaces in fractal L^2-spaces, J. Anal. Math. 75 (1998) 185–228.
[5] M. Kolountzakis, M. Matolcsi, Tiles with no spectra, preprint, arXiv: math.CA/0406127.
[6] I. Łaba, Fuglede's conjecture for a union of two intervals, Proc. Amer. Math. Soc. 129 (2001) 2965–2972.
[7] I. Łaba, Y. Wang, On spectral Cantor measures, J. Funct. Anal. 193 (2002) 409–420.
[8] J.C. Lagarias, Y. Wang, Spectral sets and factorizations of finite Abelian groups, J. Funct. Anal. 145 (1997) 73–98.
[9] H. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, Acta Math. 117 (1967) 37–52.

[10] S. Pedersen, Y. Wang, Universal spectra, universal tiling sets and the spectral set conjecture, Math. Scand. 88 (2001) 246–256.
[11] R. Strichartz, Mock Fourier series and transforms associated with certain Cantor measures, J. Anal. Math. 81 (2000) 209–238.
[12] T. Tao, Fuglede's conjecture is false in 5 and higher dimensions, Math. Res. Lett. 11 (2004) 251–258.

Further reading

[13] P.E.T. Jorgensen, S. Pedersen, Spectral pairs in Cartesian coordinates, J. Fourier Anal. Appl. 5 (1999) 285–302.
[14] M. Kolountzakis, Non-symmetric convex domains have no basis of exponentials, Illinois J. Math. 44 (2000) 542–550.
[15] I. Łaba, The spectral set conjecture and multiplicative properties of roots of polynomials, J. London Math. Soc. 65 (2002) 661–671.
[16] J.C. Lagarias, J.A. Reeds, Y. Wang, Orthonormal bases of exponentials for the n-cube, Duke Math. J. 103 (2000) 25–37.
[17] J.C. Lagarias, S. Szabo, Universal spectra and Tijdeman's conjecture on factorization of cyclic groups, J. Fourier Anal. Appl. 7 (2001) 63–70.
[18] R. Strichartz, Remarks on "Dense analytic subspaces in fractal L^2-spaces", J. Anal. Math. 75 (1998) 229–231.